JOURNAL OF APPROXIMATION THEORY 4, 442–450 (1971)

Quantitative Results for Positive Linear Approximation Operators*

ERGA CENSOR

Department of Mathematics, Technion-Israel Institute of Technology, Haifa, Israel

Communicated by Oved Shisha

Received February 25, 1971

1. INTRODUCTION

Let S be a linear space of continuous real-valued functions defined on a set K, and let $\{L_n\}_{n=1}^{\infty}$ be a sequence of positive linear operators transforming functions of S into functions of S.

A system $\{f_i\}_{i=0}^{k}$ of functions of S is said to be a system of test functions if and only if the uniform convergence of $L_n(f_i)$ to f_i $(i = 0, 1,..., k)$ implies the uniform convergence of $L_n(f)$ to f for all $f \in S$.

Korovkin [3] showed that, for the cases $S = C[a, b]$ and $S = C_{2\pi}$, such systems of test functions are $\{1, x, x^2\}$ and $\{1, \cos x, \sin x\}$, respectively. He also proved for $S = C[a, b]$ that a necessary and sufficient condition for $\{f_i\}_{i=0}^{2}$ to be a system of test functions is that (f_0, f_1, f_2) form a Chebyshev system on $[a, b]$.

In the present note we seek an estimate for $\| L_n(f) - f \|$ in terms of the corresponding quantities for the test functions.

Such estimates for $\| L_n(f) - f \|$ were derived by Shisha and Mond [6, 7] for positive linear operators defined on $S = C[a, b]$ or on $S = C_{2\pi}$. The estimates were given in terms of the corresponding numbers $\| L_n(f_i) - f_i \|$, where f_i were appropriate test functions, e.g., $\{1, x, x^2\}$ and $\{1, \cos x, \sin x\}$, respectively.

In Section 2 we generalize the results of Shisha and Mond to the m-dimensional case. 2.1 deals with polynomial test functions (Theorem 1), while 2.2 deals with the trigonometric case (Theorem 2). All the derived estimates for $\| L_n(f) - f \|$ involve a modulus of continuity $\omega(\delta) = \omega(f; \delta)$, which we define in a natural way for functions of m variables.

The set-up discussed in Section 3 involves $S = C[a, b]$ and a Chebyshev

* This paper is a part of an M.Sc. thesis submitted to the Technion, I.I.T., Haifa, Israel. The author wishes to express her deep gratitude to Professor Zvi Ziegler for his guidance and inspiration.

system on $[a, b]$, (f_0, f_1, f_2), serving as the system of test functions. Estimates for $\| L_n(f) - f \|$ in terms of $\| L_n(f_i) - f_i \|$ $(i = 0, 1, 2)$ were given by Shisha and Mond [6] for Chebyshev systems satisfying additional conditions. We show that such estimates can be obtained for general Extended Chebyshev systems (Theorem 4).

In Section 4, the approximated functions are assumed to have a continuous derivative on $[a, b]$. Theorem 5 and Theorem 7 yield estimates for $\| L_n(f) - f \|$ in terms of polynomial test functions, and in terms of test functions forming an Extended Chebyshev system, respectively.

2. Generalizations to the m-Dimensional Case

2.1. *The polynomial Case*

Let K_m be a compact subset of the Euclidean space R_m. Let $S = C_{K_m}$ be the set of all continuous real functions on K_m. It was shown by Volkov [9] that the following $m + 2$ functions are test functions for C_{K_m}:

$$\begin{aligned} f_{0m}(x_1, x_2, \ldots, x_m) &= 1, \\ f_{jm}(x_1, x_2, \ldots, x_m) &= x_j, \qquad j = 1, \ldots, m, \\ f_{m+1,m}(x_1, x_2, \ldots, x_m) &= x_1^2 + x_2^2 + \cdots + x_m^2. \end{aligned}$$

In order to extend the result of Shisha and Mond ([6], Theorem 1) to C_{K_m}, we first define $\omega(\delta)$, the modulus of continuity of $f(x_1, \ldots, x_m) \in C_{K_m}$, where K_m is also assumed to be convex.

Definition 1. The modulus of continuity $\omega(\delta)$ is defined for $f \in C_{K_m}$ by:

$$\omega(\delta) = \omega(f; \delta) = \max_{\substack{x, y \in K_m \\ d(x,y) \leq \delta}} |f(x) - f(y)|, \tag{1}$$

where $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$ and

$$d(x, y) = \left[\sum_{i=1}^{m} (x_i - y_i)^2 \right]^{1/2}. \tag{2}$$

Theorem 1. *Let K_m be a compact and convex subset of R_m, and let $L_1, L_2, \ldots$ be positive linear operators on C_{K_m}. Suppose that $\{L_n(1)\}_{n=1}^{\infty}$ is uniformly bounded in K_m. Let $\omega(\delta)$ be the modulus of continuity of $f \in C_{K_m}$; then for $n = 1, 2, \ldots$, we have*

$$\| L_n(f) - f \| \leq \| f \| \cdot \| L_n(1) - 1 \| + \| L_n(1) + 1 \| \, \omega(f; \mu_n), \tag{3}$$

where

$$\mu_n = \left\| L_n \left\{ \sum_{i=1}^{m} (\xi_i - x_i)^2; x_1, ..., x_m \right\} \right\|^{1/2} \tag{4}$$

and $\| \ \|$ *stands for the* sup norm *over* K_m. *In particular, if* $L_n(1) = 1$, (3) *reduces to*

$$\| L_n(f) - f \| \leqslant 2\omega(f; \mu_n). \tag{5}$$

(In (4), L_n operates on a function of $\xi_1, ..., \xi_m$; the resulting function is evaluated at the point $(x_1, x_2, ..., x_m)$).

The proof of our theorem resembles closely that of the corresponding theorem of Shisha and Mond and is therefore omitted.

2.2. *The trigonometric Case*

Let U be a unit circle with an angular coordinate x. We regard U as the additive group of real numbers taken modulo 2π. The distance between two points $x, t \in U$ $(0 \leqslant x, t \leqslant 2\pi)$ is

$$d_1(x, t) = \min\{| t - x |, 2\pi - | t - x |\}. \tag{6}$$

Note that we always have $d_1(x, t) \leqslant \pi$.

The distance between any two reals is the distance between their representations, as defined in (6). We identify a function $f \in C_U$ with the corresponding continuous, 2π-periodic function on $(-\infty, \infty)$ (see [4, p. 2]).

Definition 2. The trigonometric modulus of continuity is defined, for $f \in C_U$, by:

$$\omega(\delta) = \max_{\substack{x,t \in U \\ d_1(x,t) \leqslant \delta}} | f(x) - f(t) |, \tag{7}$$

where $d_1(x, t)$ is defined in (6).

We discuss now the trigonometric m-dimensional case.

Let $K = U_m$, an m-dimensional torus, be the cartesian product of m unit circles U. We identify C_{U_m} with the set of all continuous real functions $f(x_1, ..., x_m)$ which are 2π-periodic in each x_k.

Morozov [5] showed that the following $2m + 1$ functions are test functions for C_{U_m}:

$$f_{0m}(x_1, ..., x_m) = 1,$$
$$f_{jm}(x_1, ..., x_m) = \sin x_j, \qquad j = 1, ..., m,$$
$$f_{m+j,m}(x_1, ..., x_m) = \cos x_j, \qquad j = 1, ..., m.$$

We generalize now the trigonometric result of Shisha and Mond [6, Theorem 3; 7] to the case $m = 2$; the proof for $m > 2$ is similar.

DEFINITION 3. For $f \in C_{U_2}$, its trigonometric modulus of continuity is defined by:

$$\omega(\delta) = \max_{\substack{x, y \in U_2 \\ d_2(x,y) \leqslant \delta}} |f(x) - f(y)|, \tag{8}$$

where

$$d_2(x, y) = \{[d_1(x_1, y_1)]^2 + [d_1(x_2, y_2)]^2\}^{1/2}, \tag{9}$$

$x = (x_1, x_2)$, $y = (y_1, y_2)$; $d_1(x_i, y_i)$ is defined as in (6).

THEOREM 2. *Let $K = U_2 = U \times U$ be a 2-dimensional torus, U being a unit circle. Let $L_1, L_2, \ldots$ be positive linear operators on C_{U_2}. Suppose that $\{L_n(1)\}_{n=1}^{\infty}$ is uniformly bounded in K. Let $\omega(\delta)$ be the trigonometric modulus of continuity of $f \in C_{U_2}$. Then for $n = 1, 2, \ldots$, (3) holds, where*

$$\mu_n = \pi \left\| L_n \left\{\sin^2 \frac{\xi - x}{2} + \sin^2 \frac{\eta - y}{2}; x, y\right\} \right\|^{1/2} \tag{10}$$

and $\| \ \|$ stands for the sup norm *over K. In particular, if $L_n(1) = 1$, we get* (5) *with μ_n as in* (10).

Proof. Let $(\xi, \eta) \in K$, $(x, y) \in K$, and let δ be a positive number ($\leqslant \pi\sqrt{2}$). Since $d_1(\xi, x) \leqslant \pi$, $d_1(\eta, y) \leqslant \pi$, we have $d_1(\xi, x) \leqslant \pi \sin |\xi - x|/2$, $d_1(\eta, y) \leqslant \pi \sin |\eta - y|/2$. Thus, $d_2[(\xi, \eta), (x, y)] \geqslant \delta$ implies that

$$\begin{aligned}
|f(\xi, \eta) - f(x, y)| &\leqslant \omega\{d_2[(\xi, \eta), (x, y)]\} \\
&= \omega\{\sqrt{[d_1(\xi, x)]^2 + [d_1(\eta, y)]^2}\} = \omega\{\sqrt{[d_1(\xi, x)]^2 + [d_1(\eta, y)]^2}\, \delta^{-1}\delta\} \\
&\leqslant \{1 + ([d_1(\xi, x)]^2 + [d_1(\eta, y)]^2)\, \delta^{-2}\}\, \omega(\delta) \\
&\leqslant \left\{1 + \left(\frac{\pi}{\delta}\right)^2 \left(\sin^2 \frac{\xi - x}{2} + \sin^2 \frac{\eta - y}{2}\right)\right\} \omega(\delta).
\end{aligned}$$

The resulting inequality

$$|f(\xi, \eta) - f(x, y)| \leqslant \left\{1 + \left(\frac{\pi}{\delta}\right)^2 \left(\sin^2 \frac{\xi - x}{2} + \sin^2 \frac{\eta - y}{2}\right)\right\} \omega(\delta)$$

holds for any two points (ξ, η) and (x, y) because of the periodicity of f, and the fact that $\sin^2(\cdot)$ has period π. The last inequality is obviously valid if $d_2[(\xi, \eta), (x, y)] < \delta$.

Let n be a positive integer; then

$$\begin{aligned}
&| L_n\{f(\xi, \eta); x, y\} - f(x, y)\, L_n\{1; x, y\}| \\
&\quad \leqslant \omega(\delta) \left[L_n\{1; x, y\} + \left(\frac{\pi}{\delta}\right)^2 L_n \left\{\sin^2 \frac{\xi - x}{2} + \sin^2 \frac{\eta - y}{2}; x, y\right\}\right] \\
&\quad \leqslant \omega(\delta) \left[L_n\{1; x, y\} + \left(\frac{\mu_n}{\delta}\right)^2\right].
\end{aligned}$$

If $\mu_n > 0$, we take $\delta = \mu_n$, and by making use of the obvious inequality

$$| -f + fL_n(1)| \leqslant \|f\| \cdot \| L_n(1) - 1 \|, \tag{11}$$

we easily obtain (3).

If $\mu_n = 0$, then for every $\delta > 0$, $| L_n(f) - fL_n(1)| \leqslant \omega(\delta) L_n(1)$. Letting $\delta \to 0^+$, we obtain $L_n(f) = fL_n(1)$, and by (11), $|f - L_n(f)| \leqslant \|f\| \| L_n(1) - 1 \|$ which is (3) in this case. Q.E.D.

Remark. In the last proof we have used the method of the proof of Shisha and Mond, but our definition (8) has enabled us to extend their result to the m-dimensional case. Furthermore, the proof of Shisha and Mond for the one-dimensional case can be shortened by using $\omega(\delta)$ of Definition 2.

3. Results for Extended Chebyshev Systems

Shisha and Mond proved the following theorem:

THEOREM 3 [6, Theorem 2]. *Let $-\infty < a < b < \infty$, and let $\{L_n\}_{n=1}^{\infty}$ be a sequence of positive linear operators, defined on $C[a, b]$. Let $\{f_i\}_{i=0}^{2}$ be a Chebyshev system on $[a, b]$ such that for all $t, x \in [a, b]$,*

$$\begin{aligned} F(x, t) &\equiv \sum_{k=0}^{2} a_k(x) f_k(t) \geqslant K(t - x)^2, \\ F(x, x) &= 0, \end{aligned} \tag{12}$$

where the $a_k(x)$ are bounded functions on $[a, b]$ and K is a positive constant independent of x and t. Let $f \in C[a, b]$ have modulus of continuity ω. Then (3) holds, where now

$$\mu_n = \left\{ \| L_n\{F(t, x); x\} \| \cdot \frac{1}{K} \right\}^{1/2}, \tag{13}$$

and $\| \ \|$ stands for the sup norm *over $[a, b]$. If, in particular, $L_n(1) = 1$, we get* (5).

If, in addition, $f_k'(x) \in C[a, b]$ $(k = 0, 1, 2)$, the authors consider

$$F(t, x) = \pm \begin{vmatrix} f_0(x) & f_1(x) & f_2(x) \\ f_0'(x) & f_1'(x) & f_2'(x) \\ f_0(t) & f_1(t) & f_2(t) \end{vmatrix}, \tag{14}$$

where the sign is chosen so that $F(t, x) \geqslant 0$ for all $t, x \in [a, b]$.

They prove that if $f_k''(x)$ $(k = 0, 1, 2)$ exists throughout (a, b) and

$$\begin{vmatrix} f_0(x) & f_1(x) & f_2(x) \\ f_0'(x) & f_1'(x) & f_2'(x) \\ f_0''(y) & f_1''(y) & f_2''(y) \end{vmatrix} \geqslant 2K > 0$$

for every $x, y \in [a, b]$, then $F(t, x)$ of (14) satisfies (12).

Now let $\{f_i\}_{i=0}^{2}$ be an Extended Chebyshev system on $[a, b]$, i.e., f_0, f_1, and f_2 are twice continously differentiable and no nontrivial linear combination $\sum_{i=0}^{2} a_i f_i(x)$ has more than two zeros in $[a, b]$ (multiplicities being counted).

THEOREM 4. *Let $L_1, L_2, \ldots$ be positive linear operators on $C[a, b]$ such that $\{L_n(1)\}_{n=1}^{\infty}$ is uniformly bounded in $[a, b]$. Let $F(t, x)$ be defined as in* (14) *where $\{f_i\}_{i=0}^{2}$ is an Extended Chebyshev system on $[a, b]$. Let $f \in C[a, b]$ have modulus of continuity ω. Then the hypotheses and, hence, the conclusions of Theorem* 3 *hold.*

Proof. We have

$$\begin{aligned} F(t, x) &\geqslant K(t - x)^2, \\ F(x, x) &= 0, \end{aligned} \tag{15}$$

for all $t, x \in [a, b]$, where $K(>0)$ is independent of t and x. (See Freud [2, Lemma 2, p. 367.]) Q.E.D.

4. Approximation of a Function Belonging to $C^1[a, b]$

In this section we discuss quantitative estimates for the rapidity of convergence of $L_n(f)$ to f, where $\{L_n\}_{n=1}^{\infty}$ is a sequence of positive linear operators, and $f \in C^1[a, b]$. For the sake of completeness we consider, first, the case where the test functions are 1, x, x^2, although our next result (Theorem 5) has been already obtained by R. DeVore (in a forthcoming article [1]).

THEOREM 5. *Let $L_1, L_2, \ldots$ be positive linear operators on $C[a, b]$. Suppose that $\{L_n(1)\}_{n=1}^{\infty}$ is uniformly bounded in $[a, b]$. Let $f \in C^1[a, b]$ and let $\omega(f'; \cdot)$ be the modulus of continuity of f'. Then, for $n = 1, 2, \ldots$,*

$$\| L_n(f) - f \| \leqslant \|f\| \cdot \| L_n(1) - 1 \| + C_n \|f'\| \mu_n + C_n \mu_n \omega(f'; \mu_n), \tag{16}$$

where $C_n = 1 + \| L_n(1)\|^{1/2}$.

$$\mu_n = \| L_n\{(t - x)^2; x\}\|^{1/2}. \tag{17}$$

In particular, if $L_n(1) = 1$, (16) *reduces to*

$$\| L_n(f) - f \| \leqslant \| f' \| \mu_n + 2\mu_n \omega(f'; \mu_n). \tag{18}$$

If, in addition, $L_n\{t; x\} \equiv x$, *we obtain*

$$\| L_n(f) - f \| \leqslant 2\mu_n \omega(f'; \mu_n). \tag{19}$$

Proof. For any $x, t \in [a, b]$, there exists a point ξ between them such that

$$f(t) - f(x) = (t - x) f'(x) + (t - x)[f'(\xi) - f'(x)], \tag{20}$$

$$\begin{aligned} | f'(\xi) - f'(x) | &\leqslant \omega(f'; | \xi - x |) \leqslant \omega(f'; | t - x |) \\ &= \omega(f'; | t - x | \delta^{-1}\delta) \leqslant (1 + | t - x | \delta^{-1})\, \omega(f'; \delta). \end{aligned} \tag{21}$$

Using (20), (21), and the inequalities

$$| L_n(f) | \leqslant L_n(| f |) \qquad \text{and} \qquad L_n(f \cdot g) \leqslant \sqrt{L_n(f^2)\, L_n(g^2)},$$

we obtain

$$\begin{aligned} &| L_n(f) - fL_n(1) | \\ &\quad \leqslant | f'(x)\, L_n(t - x) | + L_n\{| t - x | \cdot | f'(\xi) - f'(x) |\} \\ &\quad \leqslant | f'(x) |\, L_n(| t - x |) + \omega(f'; \delta)\, L_n\{| t - x | [1 + | t - x | \delta^{-1}]\} \\ &\quad = | f'(x) |\, L_n(| t - x |) + \omega(f'; \delta)[L_n(| t - x |) + L_n\{(t - x)^2\}\, \delta^{-1}] \\ &\quad \leqslant | f'(x) |\, [L_n\{(t - x)^2\} \cdot L_n(1)]^{1/2} \\ &\qquad + \omega(f'; \delta)[(L_n\{(t - x)^2\} \cdot L_n(1))^{1/2} + \delta^{-1} L_n\{(t - x)^2\}]. \end{aligned}$$

Choosing $\delta = \mu_n$, we obtain

$$| L_n(f) - fL_n(1) | \leqslant \| f' \| \mu_n \| L_n(1) \|^{1/2} + (1 + \| L_n(1) \|^{1/2})\, \mu_n \omega(f'; \mu_n).$$

As before, we deduce from this inequality (16). If $\mu_n = 0$, (16) follows from (3). If $L_n(1) = 1$, we obtain (18). From the proof we see that if, in addition, $L_n\{t; x\} \equiv x$, (19) holds. Q.E.D.

THEOREM 6. *If, in addition to the hypotheses of Theorem* 5, $f \in C^{(2)}[a, b]$, *then, for* $n = 1, 2, \ldots$,

$$\| L_n(f) - f \| \leqslant \| f \| \cdot \| L_n(1) - 1 \| + C_n \mu_n (\| f' \| + \mu_n \| f'' \|). \tag{22}$$

If, in particular, $L_n(1) = 1$, (22) *reduces to*

$$\| L_n(f) - f \| \leqslant \mu_n (\| f' \| + 2\mu_n \| f'' \|). \tag{23}$$

If, in addition, $L_n\{t; x\} \equiv x$, *then*

$$\| L_n(f) - f \| \leqslant 2\mu_n^2 \| f'' \|. \tag{24}$$

Proof. If $f \in C^{(2)}[a, b]$, then $\omega(f'; \delta) \leqslant \| f'' \| \delta$, and (22) follows from (16). In the same way (23) follows from (18), and (24) from (19). Q.E.D.

We mention that Stancu [8] has obtained, independently, similar results for discrete positive linear operators of a certain type.

The following theorem yields an estimate for $\| L_n(f) - f \|$, where $f \in C^1[a, b]$, in terms of $\| L_n(f_i) - f_i \|$, where the test functions f_0, f_1, f_2 form an Extended Chebyshev system on $[a, b]$.

THEOREM 7. *Let* $F(t, x)$ *be defined as in* (14), *where* $\{f_i\}_{i=0}^{2}$ *is an Extended Chebyshev system on* $[a, b]$. *Let* $L_1, L_2, \ldots$ *be positive linear operators on* $C[a, b]$ *such that* $\{L_n(1)\}_{n=1}^{\infty}$ *is uniformly bounded in* $[a, b]$. *Then* (16) *holds, where now* μ_n *is defined as in* (13) (*cf.* (15)). *If, in particular,* $L_n(1) = 1$, *then* (18) *holds.*

Proof. In the course of proving Theorem 5 we obtained the following intermediate result:

$$| L_n(f) - fL_n(1) | \leqslant | f'(x) | \, [L_n\{(t - x)^2\} \cdot L_n(1)]^{1/2} + \omega(f'; \delta)[(L_n\{(t - x)^2\} \cdot L_n(1))^{1/2} + \delta^{-1} L_n\{(t - x)^2\}].$$

Using the fact that $F(t, x) \geqslant K(t - x)^2$, we obtain

$$| L_n(f) - fL_n(1) | \leqslant | f'(x) | \left[\frac{1}{K} L_n\{F(t, x)\} \cdot L_n(1)\right]^{1/2} + \omega(f'; \delta) \left[\left(\frac{1}{K} L_n\{F(t, x)\} \cdot L_n(1)\right)^{1/2} + \delta^{-1} L_n\{F(t, x)\} \cdot \frac{1}{K}\right].$$

Choosing $\delta = \mu_n$, we deduce

$$| L_n(f) - fL_n(1) | \leqslant \| f' \| \, \mu_n \, \| L_n(1) \|^{1/2} + (1 + \| L_n(1) \|^{1/2}) \, \mu_n \omega(f'; \mu_n)$$

and this implies (16). (Again, if $\mu_n = 0$, (16) follows from (3).) From the proof we find that if $L_n(1) = 1$, (18) holds. Q.E.D.

Remark. If $f \in C^{(2)}[a, b]$, then the conclusions of Theorem 6 hold, with μ_n defined by (13).

References

1. R. DeVore, Optimal convergence of positive linear operators, to appear.
2. G. Freud, On approximation by positive linear methods, II., *Studia Sci. Math. Hungar.* **3** (1968), 365–370.
3. P. P. Korovkin, "Linear Operators and Approximation Theory," Hindustan Publishing Corp., Delhi, 1960.
4. G. G. Lorentz, "Approximation of Functions," Holt, Rinehart, and Winston, New York, 1966.
5. E. N. Morozov, Convergence of a sequence of positive linear operators in the space of continuous 2π-periodic functions of two variables, *Kalinin. Gos. Ped. Inst. Učen. Zap.* **26** (1958), 129–142 (Russian); MR **24**, # A 957.
6. O. Shisha and B. Mond, The degree of convergence of sequences of linear positive Operators, *Proc. Nat. Acad. Sciences U.S.A.* **60** (1968), 1196–1200.
7. O. Shisha and B. Mond, The degree of approximation to periodic functions by linear positive operators, *J. Approximation Theory* **1** (1968), 335–339.
8. D. D. Stancu, Use of probabilistic methods in the theory of uniform approximation of continuous functions, *Rev. Roumaine Math. Pures Appl.*, Tome XIV, No. 5 (1969), 673–691.
9. V. I. Volkov, On the convergence of sequences of linear positive operators in the space of continuous functions of two variables, *Dokl. Akad. Nauk SSSR* **115** (1957), 17–19 (Russian); MR **20**, #1205.